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PERIODIC SOLUTIONS OF CERTAIN NONLINEAR AUTONOMOUS SYSTEMS

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The structure of the period of a parametric solution of a certain nonlinear autonomous system (which is in a sense a generalization of Liapunov systems) is investigated. The existence of a periodic solution is due to the existence of the necessary number of first integrals. Formulas for approximate calculation of the period are derived for cases where such a solution can be said to exist. The results can be applied to the study of periodic solutions of systems close to that analyzed here under principal-resonance conditions (in the sense of Malkin).

1. Formulation of the problem. We consider the system

$$dx_i / dt = a_{i1}x_1 + \ldots + a_{in}x_n + X_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{1.1}$$

where a_{ij} are constants and X_i are analytic nonlinear functions of the variables $x_1, \ldots, x_n$.

Let us assume that Eq.

$$|a_{ij} - \delta_{ij}\rho| = 0 \tag{1.2}$$

has l zero roots associated with l groups of solutions, two roots $\pm \lambda\sqrt{-1}$, and no roots which are multiples of $\pm \lambda\sqrt{-1}$.

Applying a linear nonsingular transformation with constant coefficients, we transform system (1.1) into [1 and 2]

$$du_j / dt = U_j, \quad dy / dt = -\lambda z + Y, \quad dz / dt = \lambda y + Z$$

$$dv_i / dt = b_{i1}v_1 + \ldots + b_{im}v_m + V_i \tag{1.3}$$

$$(j = 1, \ldots, l; \; i = 1, \ldots, m, \; m + l + 2 = n)$$

where U_j, Y, Z, V_i are analytic nonlinear functions of the variables $u_1, \ldots, u_i, y, z, v_1, \ldots, v_m$, and where the constants b_{ij} are such that there are no zero roots or multiples of $\pm \lambda\sqrt{-1}$ among the roots of the equation $|b_{ij} - \delta_{ij}\rho| = 0$.

Let us assume that system (1.3) has $l + 1$ analytic first integrals

$$M_j(u) + M_j^{(1)}(u, y, z, v) = C_j \qquad (j = 1, \ldots, l) \tag{1.4}$$

$$y^2 + z^2 + \psi(u, y, z, v) = C_{l+1} \tag{1.5}$$

where M_j are linear independent forms of the variables $u_1, \ldots, u_i$; $M_j^{(1)}$, ψ are terms nonlinear in u, y, z, v. Differentiating integral (1.5) and recalling (1.3), we find that first-order terms in ψ can occur only as of the quadratic forms $L_1(u)$, $L_2(v)$ with constant coefficients.

Since the forms M_j $(j = 1,\ldots, l)$ are independent, we can always take $M_j \equiv u_j$. This allows us to rewrite integrals (1.4) as

$$u_j + \varphi_j(u, y, z, v) = C_j \quad (j = 1,\ldots, l) \tag{1.6}$$

where φ_j are terms nonlinear in u, y, z, v.

Provided integrals (1.5), (1.6) exist, autonomous system (1.3) has a periodic solution in the neighborhood of the origin; this solution depends on $l + 2$ parameters [1 and 3]. Let us take the initial values of the critical variables $u_1,\ldots, u_l$, y, z as these parameters. (This is always possible [2]).

Let us investigate the structure of the period of the transformed solution of system (1.3). The above results are necessary for the investigation of periodic solutions of systems close to (1.3) under principle-resonance conditions [2], when we need to know at least the lower-order terms in the expansion of the period.

2. The general expression of the period. The only case which we shall analyze in detail is that where Eq. (1.2) has a single zero root $(l = 1)$. Our argument can be readily extended to the case $l > 1$ (see the Remark in Sect. 4).

We denote system (1.3) and integrals (1.5), (1.6) for $l = 1$ by (1.3^1), (1.5^1), (1.6^1) (system A^1), respectively.

Let us assume that the system (A^1) has already been transformed in accordance with the following statement.

Lemma 2.1. System (A^1) can always be transformed in such a way that the functions U, Y, Z, V_i, φ vanish for $y = z = v_1 = \ldots = v_m = 0$.

The validity of the lemma follows from analysis of the special case of a single zero root [1].

Proof. Let us solve Eqs.

$$-\lambda z + Y = 0, \qquad \lambda y + Z = 0, \qquad b_{i1}v_1 + \ldots + b_{im}v_m + V_i = 0$$

for $y_*(u)$, $z_*(u)$, $v_{i*}(u)$ and make the followinf substitutions in (A^1):

$$y = y_1 + y_*(u), \qquad z = z_1 + z_*(u), \qquad v_k = v_k^{(1)} + v_{k*}(u) \tag{2.1}$$

The identity obtained by differentiating transformed integral (1.6^1) and recalling (2.1) implies that the new nonlinear terms U_1, Y_1, Z_1, $V_i^{(1)}$ vanish for $y_1 = z_1 = v_1^{(1)} = \ldots = v_m^{(1)} = 0$.

Since the transformed system has the solution

$$u = C_1, \qquad y_1 = z_1 = v_1^{(1)} = \ldots = v_m^{(1)} = 0 \tag{2.2}$$

it follows that $\varphi_1(u, 0,\ldots, 0) = 0$. Transformed integral (1.5^1) now implies that solution (2.2) is associated with the following value of the constant C_2:

$$C_2 = \psi_1(C_1, 0,\ldots, 0)$$

Let us eliminate the variable u from (A^1) and apply integral (1.6^1). This yields the system

$$dy / dt = -\lambda z + Y_*, \quad dz / dt = \lambda y + Z_*$$
$$dv_i / dt = b_{i1}v_1 + \ldots + b_{im}v_m + V_{i*} \quad (i = 1, \ldots, m) \tag{2.3}$$

and the corresponding integral

$$y^2 + z^2 + \psi_* = C_3, \quad C_3 \equiv C_2 - \psi(C_1, 0,\ldots, 0) \tag{2.4}$$

The asterisks in (2.3), (2.4) denote functions analytic in $y,\ z,\ v_1,\dots,\ v_m,\ C_1$ which vanish (by Lemma 2.1) for $y = z = v_1 = \dots = v_m = 0$.

The functions $Y_*,\ Z_*,\ V_{i*}$ can contain first-order terms in $y,\ z,\ v_1,\dots,\ v_m$ whose coefficients are analytic functions of C_1 which vanish for $C_1 = 0$. We can make the following statement concerning the form of the function ψ_*:

Lemma 2.2. For sufficiently small values of the constant C_1 the function ψ_* in (2.4) does not contain first-order terms in $y,\ z,\ v_1,\dots,\ v_m$.

Proof. Let

$$\psi_* = Ay + Bz + D_1 v_1 + \dots + D_m v_m + \dots \tag{2.5}$$

where the nonlinear terms have not been written out.

Let us differentiate integral (2.4) in accordance with system (2.3). Equating the coefficients of the first powers of $y, z, v_1, \dots, v_m$ in the resulting identity to zero, we obtain a homogeneous system for determining the unknowns $A, B, D_1,\dots, D_m$. The determinant of this system is a continuous function of C_1 which becomes the nonzero quantity $\lambda^2|b_{ij}|$ for $C_1 = 0$. Then for sufficiently small C_1 we have $A = B = D_1 = \dots = D_m = 0$ in (2.5).

Making use of Lemmas 2.1 and 2.2, we can obtain the expression for the period of the solution of system (2.3), (2.4), and hence for the period of the solution of system (A^1) in the usual way [1 and 2]. Let us outline the derivation of the formula for the period, omitting the detailed expressions given in [1 and 2].

We begin by setting

$$y = \rho\cos\vartheta, \qquad z = \rho\sin\vartheta, \qquad v_i = \rho\chi_i \qquad (i = 1,\dots,\ m) \tag{2.6}$$

in (2.3), (2.4).

For sufficiently small ρ, χ we can set $C_3 = C^2$ in (2.4) and (recalling Lemma 2.2) solve integral (2.4) for ρ

$$\rho = \pm C\,[1 + G_1(C_1,\ \chi,\ \vartheta) + CG_2(C,\ C_1,\ \chi,\ \vartheta)] \tag{2.7}$$

where $G_1,\ G_2$ are analytic functions of their arguments and are periodic in ϑ; $G_1(0,\ 0,\ \vartheta) = 0$.

Making use of (2.7), we can replace system (2.3) by the mth order system

$$\frac{d\chi_i}{d\vartheta} = \frac{1}{\lambda}\sum_{j=1}^{n}(b_{ij} + d_{ij}(C_1,\ \vartheta))\,\chi_j + C_1K_i^{(1)}(C_1,\ \vartheta) + CK_i^{(2)}(C,\ \chi,\ C_1,\vartheta)$$

$$(i = 1,\ \dots,\ m) \tag{2.8}$$

where $d_{ij}\ (d_{ij}(0,\ \vartheta) = 0),\ \ K_i^{(1)},\ \ K_i^{(2)}$ are analytic functions of their arguments and are periodic in ϑ.

For sufficiently small values of the constants C_1, C system (2.8) has a periodic solution of the form [2]

$$\chi_i = \chi_i(C_1,\ C,\ \vartheta), \qquad \chi_i(0,\ 0,\ \vartheta) = 0 \qquad (i = 1,\ \dots,\ m) \tag{2.9}$$

The following formula is then valid for the period of the solution of system (2.8), and therefore for the period of the solution of system (A^1) (see [1], p. 249):

$$T = \int_0^{2\pi}\frac{\rho\, d\vartheta}{\lambda\rho + (Z_*)\cos\vartheta - (Y_*)\sin\vartheta} \tag{2.10}$$

where the parentheses indicate successive substitutions in accordance with (2.6), (2.7), (2.9).

From (2.10) we obtain the period

$$T = 2\pi\lambda^{-1}[1 + p(C_1) + q(C_1,\ C)]$$

$$p(C_1) = \sum_{i\geqslant 1} p_i C_1^i, \qquad = \sum_{j\geqslant 1}\sum_{k\geqslant 0} q_{2j,k} C_1^k C^{2j} \tag{2.11}$$

where p_i, $q_{2j,k}$ are constants, and where C occurs in even powers only [1].

Recalling integrals (1.6), (2.4) and setting

$$C_1 = a + \varphi(a, b, c), \quad C^2 \equiv C_2 - \psi(C_1, 0, \ldots, 0), \quad C_2 = b^2 + c^2 + \psi(a, b, c) \qquad (2.12)$$

in (1.6) (here a, b, c are the initial values of the variables u, y, z), we obtain the required expression for the period $T(a, b, c)$ of the solution of system (A^1).

We note that the expansion of the period $T(a, b, c)$ can begin with terms of any order (the odd order is introduced by the constant C_1).

From now on (Sects. 3, 4) we shall consider system (A^1) without the variables $v_1, \ldots, v_m$

$$du / dt = U(u, y, z), \qquad dy / dt = -\lambda z + Y(u, y, z), \qquad dz / dt = \lambda y + Z(u, y, z) \qquad (2.13)$$

with the integrals

$$u + \varphi(u, y, z) = C_1 \quad \text{(a)}, \qquad y^2 + z^2 + \psi(u, y, z) = C_2 \quad \text{(b)} \qquad (2.14)$$

This does not reduce the generality of the problem, since system (A^1) can always be reduced to the form (2.13), (2.14) (e.g. see [2]).

Let us set out the rules for computing the lower-order terms in the expansion of the period $T(a, b, c)$.

3. The function $p(C_1)$ in (2.11). Let us consider the integrand in (2.10) after making substitutions (2.6), (2.7), (2.9) and dividing through by ρ in the numerator and denominator.

The function $p(C_1)$ in (2.11) is due to the presence of terms linear in y, z in Y_*, Z_* (system (2.3) without the variables v_j) which in turn consist of products of the form $u^i y$, $u^i z$ $(i \geqslant 1)$ in Y, Z (system (2.13) after the elimination of u by means of integral (2,14a)).

Let us assume that the functions Z, Y contain the terms

$$\alpha_i u^i y + \beta_i u^i z, \qquad \gamma_i u^i y + \delta_i u^i z \quad (i \geqslant 1) \qquad (3.1)$$

respectively.

After the necessary operations, expressions (3.1) yield the following value of the integrand in (2.10):

$$\{\lambda + [\alpha_i \cos^2 \vartheta + (\beta_i - \gamma_i) \sin \vartheta \cos \vartheta - \delta_i \sin^2 \vartheta] C_1^i\}^{-1} \qquad (3.2)$$

where we have omitted terms dependent on C and also terms dependent on powers of C_1 other than i.

Dividing in (3.2) and integrating from 0 to 2π, we obtain the value of $p(C_1)$ in (2.11).

All this implies the validity of the following statement.

Lemma 3.1. a) Let no terms of the form (3.1) be present in the functions Y, Z (system (2.13)); alternatively, if such terms are present, let Y contain only $\gamma_i u^i y$, and Z only $\beta_i u^i z$. In the latter case let $\gamma_i = \beta_i$ for all i. This means that $p(C_1) \equiv 0$ in (2.11).

b) Let Z contain the term $\alpha_h u^h y$, and Y the term $\delta_h u^h z$, $\alpha_h \neq \delta_h$, and let one of the conditions of Par. (a) above hold for all the terms (3.1) in Y, Z for $1 \leqslant i < h$. Then

$$p(C_1) = (2\lambda)^{-1} (\delta_h - \alpha_h) C_1^h + (\cdots) C_1^{h+1} + \ldots \qquad (3.3)$$

c) If Z contains the term $\beta_\sigma u^\sigma z$, and Y the term $\gamma_\sigma u^\sigma y$, $\beta_\sigma \neq \gamma_\sigma$ and let one of the requirements of Par. (a) be fulfilled for all terms (3.1) in Y, Z for $1 \leqslant i < \sigma$, $\sigma < i \leqslant 2\sigma$. Then

$$p(C_1) = (8\lambda^2)^{-1} (\beta_\sigma - \gamma_\sigma)^2 C_1^{2\sigma} + (\cdots) C_1^{2\sigma+1} + \ldots \qquad (3.4)$$

4. The function $q(C_1, C)$ in (2.11). We shall limit ourselves to the computation of the coefficients $q_{2k,0}$ $(k \geqslant 1)$. These coefficients consist solely of terms which are free in C_1 in expression (2.7) for ρ. Let us consider a particular case.

Let the functions φ, ψ in (2.14) vanish for $u=0$. Differentiating integrals (2.14) in accordance with system (2.13), we find that $U(0, y, z)=0$, and that the functions Y, Z can contain terms independent of u only if they satisfy Eq.

$$2yY + 2zZ = 0 \tag{4.1}$$

Let Y contain the terms

$$-s_1 z^i y^j - s_2 z^{i+1} y^{j-1} \tag{4.2}$$

and Z the terms

$$s_1 z^{i-1} y^{j+1} + s_2 z^i y^j \tag{4.3}$$

satisfying condition (4.1).

In this particular case the functions ψ_* in (2.4) and G_1, G_2 in (2.7) (without the variables v_j, χ_j, respectively) vanish for $C_1 = 0$.

Expressions (4.2), (4.3) give us the value of the integrand in (2.10).

$$[\lambda + (s_1 \sin^{i-1}\vartheta \cos^j\vartheta + s_2 \sin^i\vartheta \cos^{j-1}\vartheta)\, C^{i+j-1}]^{-1} \tag{4.4}$$

where we have omitted terms which depend on C_1 and also terms which depend on powers of C different from $i+j-1$.

Dividing in (4.4) and integrating over ϑ from 0 to 2π, we obtain the value of the coefficient $q_{2k,0}$ in (2.11). This means that the following lemma is valid.

Lemma 4.1. a) Let the functions φ, ψ, Y, Z in (2.13), (2.14) vanish for $u=0$. Then $q_{2k,0} = 0$ $(k = 1, 2, \ldots)$ in (2.11).

b) Let the functions φ, ψ vanish for $u=0$ and let the lowest-order terms in the functions Y, Z be (4.2), (4.3). The first nonzero coefficient $q_{2k,0}$ in (2.11) is then given by

$$q_{i+j-1,0} = -\frac{s_1}{2\pi\lambda}\int_0^{2\pi} \sin^{i-1}\vartheta \cos^j\vartheta \, d\vartheta \tag{4.5}$$

if i is odd and j is even;

$$q_{i+j-1,0} = -\frac{s_2}{2\pi\lambda}\int_0^{2\pi} \sin^{i}\vartheta \cos^{j-1}\vartheta \, d\vartheta \tag{4.6}$$

if i is even and j is odd;

$$q_{2(i+j-1),0} = \frac{1}{2\pi\lambda^2}\int_0^{2\pi} (s_1^2 \sin^{2(i-1)}\vartheta \cos^{2j}\vartheta + s_2^2 \sin^{2i}\vartheta \cos^{2(j-1)}\vartheta)\, d\vartheta \tag{4.7}$$

if $(i+j)$ is even.

Corollary. If in attempting to use Formulas (4.5), (4.6) we find that $s_1 = 0$ or $s_2 = 0$, respectively, then the first nonzero coefficient $q_{2(i+j-1),0}$ can be found from (4.7) by setting $s_1 = 0$ or $s_2 = 0$.

Remark. Let Eq. (1.2) have $l > 1$ zero roots. Lemmas 2.1, 2.2 remain valid. Terms (3.1) must be considered for each individual variable $u_1, \ldots, u_l$ in order to construct the complete expression (3.2). Dividing in (3.2) and integrating over ϑ from 0 to 2π, we obtain the function $p(C_1, \ldots, C_l)$ in (2.11). The function q in (2.11) depends on $C_1, \ldots, C_l, C$. The condition of Lemma 4.1 must now include the requirement that the functions $\varphi_1, \ldots, \varphi_l, \psi, Y, Z$ (or only $\varphi_1, \ldots, \varphi_l, \psi$) vanish for $u_1 = \ldots = u_l = 0$.

Let us now consider some examples to illustrate how the results of Sects. 3, 4 can be used to find the lower-order terms in the expansion of the period $T(a, b, c)$. As already noted in Sect. 1, it is necessary to know these terms in order to investigate systems closely resembling (1.3).

5. Examples. 1. The system

$$du/dt = 2\lambda yz^2 (1 + u) - \lambda y^3, \quad dy/td = -\lambda z - \lambda yz + \frac{1}{2}\lambda y^2z^2$$

$$dz / dt = \lambda y - \lambda z^3 \qquad (5.1)$$

has the integrals

$$u + y^2 z = C_1, \quad y^2 + z^2 + uz^2 = C_2$$

Applying formula (2.11) and (3.3)(where $\delta_h = -\lambda$, $\alpha_h = 0$, $h = 1$) and making the substitution $C_1 = a + \ldots$ according to (2.12), we obtain

$$T = 2\pi\lambda^{-1} (1 - {}^1/_2\, a + \ldots)$$

2. The system

$$du / dt = uz, \quad dy / dt = -z - yz, \quad dz / dt = y + y^2 - {}^3/_2 u^3 \qquad (5.2)$$

has the integrals

$$u + uy = C_1, \quad y^2 + z^2 + u^3 = C_2 \qquad (5.3)$$

Let us transform system (5.2),(5.3) in accordance with Lemma 2.1. To this end we solve Eqs.

$$-z - yz = 0, \quad y + y^2 - {}^3/_2\, u^3 = 0$$

for y, z as functions of u. This yields

$$z_* (u) \equiv 0, \quad y_* (u) = {}^3/_2 u^3 + \ldots$$

Making the substitutions $z = z_* (u) + z_1$, $y = y_* (u) + y_1$ in (5.2),(5.3), we obtain

$$dy / dt = uz_1, \quad dy_1 / dt = -z_1 - y_1 z_1, \quad dz_1 / dt = y_1 + y_1^2 - 3u^3 y_1$$

$$u + uy_1 = C_1, \quad y_1^2 + z_1^2 + 3u^3 y_1 + u^3 = C_2$$

From formula (3.3) we find that $p (C_1) = {}^3/_2\, C_1^3 + \ldots$, and from formula (4.7), where $i = j = 1$, $s_1 = 1$, $s_2 = 0$, that $q_{20} = {}^1/_2$, $q (C_1, C) = {}^1/_2\, C^2 + \ldots$

Equating the functions $p (C_1)$ and $q (C_1, C)$ and making substitution (2.12), we obtain

$$T = 2\pi\, [1 + {}^1/_2\, (b^2 + c^2) + \ldots]$$

3. The system

$$du / dt = -y^2 + z^2, \qquad dy / dt = -z + {}^3/_2\, u^2 y, \qquad dz / dt = y - {}^3/_2\, u^2 z \qquad (5.4)$$

has the integrals

$$u + yz = C_1, \quad y^2 + z^2 + u^3 = C_2$$

From formula (3.4), where $\gamma_\sigma = {}^3/_2$, $\beta_\sigma = -{}^3/_2$, $\sigma = 2$, we obtain $p (C_1) = {}^9/_8 C_1^4 + \ldots$ Expression (2.11) yields $q = -{}^3/_4\, C_1 C^2 + \ldots$. Equating the functions $p (C_1)$, $q (C_1, C)$ and making substitution (2.12), we obtain

$$T = 2\pi\, [1 - {}^3/_4\, a\, (b^2 + c^2) + \ldots]$$

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